A Flute-like Plasma Instability Driven by Field Inhomogeneities

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We investigate the stability of ring-like plasma configurations with small Larmor radii, precessing axi-symmetrically in mirror machines. If the precession frequency increases with radius, such equilibria may be subject to a low frequency flute-like instability which is driven by field inhomogeneities rather than by the density gradient. The phase velocity of the unstable waves is twice that occurring in the case of the ordinary flute-instability. Though the physical situation is quite different from that of circular beams of charged particles, the driving mechanism shows a certain analogy to that of the negative mass instability. It is noteworthy, however, that in our case equilibria with a finite spread in the transverse particle energies may be less stable than those with zero energy spread.

I. Introduction

In this paper we wish to show that certain plasma configurations are subject to flute-like low frequency instabilities which are driven by the field inhomogeneities rather than by the density gradient. We shall consider especially axi-symmetric mirror configurations characterized by a relative maximum of the radial density profile at some finite value of the radius r. The plasma concentration is then particularly high in a ring-like region around the axis of the machine. Such equilibria are encountered in several mirror machines with high energy particle injection, e. g. in the M.M.I.I. device [1], in Ogra II [2], and, with a less steep density maximum, in Phoenix [3].

Some aspects of the stability of such configurations may be elaborated on adopting a simple model which is based on the following assumptions:

1) A circular plasma ring is situated axi-symmetrically in a mirror machine. It may have a space charge giving rise to a radial electric field. The ring may surround another axi-symmetric plasma in the middle of the machine which may have two effects on the ring plasma:
   a) it is dielectric,
   b) it has a space charge giving rise to an additional radial electric field.

2) The plasma pressure is small compared with the magnetic pressure.

3) The minor radius a of the ring is small compared with its major radius R.

4) The mean Larmor radius of the ions is small compared with the characteristic lengths over which the magnetic and electric fields change. Further the cyclotron frequency of the ions is large compared with the characteristic fluctuation time of the electro-magnetic field.

5) The ion velocity component $v_\parallel$ parallel to the magnetic field is negligible.

6) The electrons have negligible mass and negligible temperature.

7) In the plasma region the dependence of the magnetic field on the axial coordinate z is negligible.

The fact that such systems are unstable under certain conditions, can be seen qualitatively from a simplified picture. We shall describe it for the case that the equilibrium drift of the ions around the axis of the machine is primarily due to the gradient of the magnetic field.

Suppose that the angular drift velocity increases with radius. Consider then an initial electrostatic perturbation built up by an accumulation of ions somewhere on the ring. The azimuthal electric field of this perturbation will give rise to radial drifts of the guiding centers which have to be added to the azimuthal equilibrium precession drift. Ions behind the charge accumulation will drift outward and enter regions with higher precession frequency, so that

[1] F. Prevot et al., in Plasma Physics and Controlled Nuclear Fusion Research (Proc. Conf. Culham, 1965) 2, IAEA, Vienna 1966.

[2] L. I. Artemenko et al., in Plasma Physics and Controlled Nuclear Fusion Research (Proc. Conf. Culham, 1965) 2, IAEA, Vienna 1966. — Second Eur. Conf. Contr. Fus. Plasma Phys., Stockholm 1967.

[3] E. G. Murphy et al., CLM-P 94 [1966].

they catch up with the charge accumulation after some time. Ions in front of the perturbation will drift inward, enter regions with lower precession frequency and will be caught up by the perturbation. As a result the perturbation increases and yields instability. Of course, this consideration does not take account of the dispersive effect of the velocity shear in the ring, which may lead to stability for low plasma densities.

From the above qualitative picture it can be seen that the electrons do not play an important role in the instability mechanism. Due to their zero mass and temperature they may be thought to be fixed in space. They only feel the electric field of the unstable waves which passes by quickly with the ion precession velocity and cancels out in the average.

In the opposite case, when the precession drift is mainly due to the radial electric field, the electrons generally play a subaltern role as well. This is so because the electric field is caused by a large ion excess. In mirror machines with high energy particle injection the ion density may exceed the electron density by more than a factor of two, so that the electrons are not determinant because of their small number.

II. Formulation of a One-dimensional Model

The above consideration suggests to treat the electrons as a negatively charged background and to apply particle dynamics only to the ions. In view of assumption 4) we start from the guiding center equation of motion for the ions [4] which is valid to first order in the small parameter ϱ/L, where ϱ is the Larmor radius of the considered particle, and L is a characteristic length over which fields change. After making use of assumption 5) and neglecting the time variation of the magnetic field [considering only electrostatic perturbations due to assumptions 2) and 4)] this equation reads

$$\dot{\boldsymbol{r}} = \frac{\boldsymbol{B}}{B^2} \times \left\{ -c\,\boldsymbol{E} + \frac{\mu\,c}{e}\,\nabla B + \frac{m\,c^2}{e\,B^2}\frac{\partial \boldsymbol{E}}{\partial t} \times \boldsymbol{B} + \frac{m\,c^3}{e}\left(\frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2}\cdot\nabla\right)\frac{\boldsymbol{E}\times\boldsymbol{B}}{B^2} \right\} \tag{1}$$

where $\boldsymbol{r}$ is the position vector of the guiding center, $\boldsymbol{B}$ the magnetic field, $\boldsymbol{E}$ the electric field, c the velocity of light, e the charge of the ion, μ its magnetic moment, and m its mass.

We now introduce cylindrical coordinates r, φ, z, and make use of the fact that $\boldsymbol{B}$ is in the z-direction and depends only on r. Writing the electric field as a sum of the equilibrium part and a small perturbation,

$$E_r = E(r) + \hat{E}_r(r, \varphi, t), \qquad E_\varphi = \hat{E}_\varphi(r, \varphi, t)$$

we get after linearization in the perturbing quantities

$$\dot{r} = \left[\frac{c}{B} + \frac{m\,c^3}{e\,r\,B^2}\frac{d}{dr}\left(\frac{r\,E}{B}\right) \right]\hat{E}_\varphi + \frac{m\,c^2}{e\,B^2}\frac{\partial \hat{E}_r}{\partial t} - \frac{m\,c^3}{e\,r\,B^3}\hat{E}\frac{\partial \hat{E}_r}{\partial \varphi}, \tag{2}$$

$$\dot{\varphi} = \frac{c}{r\,B}\left[\frac{\mu}{e}\frac{dB}{dr} - E - \hat{E}_r - \frac{m\,c^2\,E^2}{e\,r\,B^2} - \frac{2\,m\,c^2\,E}{e\,r\,B^2}\hat{E}_r + \frac{m\,c}{e\,B}\frac{\partial \hat{E}_\varphi}{\partial t} - \frac{m\,c^2\,E}{e\,r\,B^2}\frac{\partial \hat{E}_\varphi}{\partial \varphi} \right]. \tag{3}$$

Since the instability we are investigating is based on particle drifts, we expect from physical reasons that it will occur in the low frequency region. This would imply that in Eqs. (2) and (3) several terms would be negligible with respect to others. A proof of the low frequency character can be given on taking account of all terms in the above equations and showing that the dispersion relation derived from these complete equations has only low frequency solutions [5]. In this paper we use this fact from the beginning in Eqs. (2) and (3).

Denoting by ω_c the cyclotron frequency, by ω a characteristic frequency of the instability, by λ its wave-length, and by v_E the drift velocity due to the equilibrium electric field, we find that the last two terms in Eq. (2) are smaller than the first one by factors ω/ω_c and $v_E/\lambda\,\omega_c$, respectively. In view of assumption 4) we must exclude wave-lengths λ which are much smaller than R. Therefore, these two terms may be neglected. In Eq. (3) the last four terms are smaller than the second or third one by factors $v_E/R\,\omega_c$, and $v_E/R\,\omega_c$, ω/ω_c, $v_E/\lambda\,\omega_c$, re-

[4] T. G. NORTHROP, The Adiabatic Motion of Charged Particles, Interscience Publishers, New York 1963.

[5] D. VOSLAMBER, EUR-CEA-FC-385 [1966].

spectively. They therefore may be neglected for the same reasons.

Our further treatment will be based essentially on assumption 3) which suggests to establish a one-dimensional formalism by projecting the particle motion onto the circular axis of the plasma ring. The procedure will be similar to that used in the theory of the negative mass instability, though the physical situation is there quite different [6-8].

We first take the time derivative of Eq. (3) (after neglecting the last terms as discussed above) and reexpress the factors $\dot{r}$ and $\dot{\varphi}$ from Eqs. (2) and (3). Retaining, again, only the linear terms in the perturbing fields $\hat{E}_r$ and $\hat{E}_\varphi$, we get

$$\ddot{\varphi} = \frac{d\dot{\varphi}_0}{dr}\left[\frac{c}{B} + \frac{m\,c^3}{e\,r\,B^2}\frac{d}{dr}\left(\frac{r\,E}{B}\right)\right]\hat{E}_\varphi - \frac{c}{r\,B}\left[\frac{\partial\hat{E}_r}{\partial t} + \frac{\partial\hat{E}_r}{\partial\varphi}\dot{\varphi}_0\right]$$

where   (4)

$$\dot{\varphi}_0 = \frac{c}{r\,B}\left(\frac{\mu}{e}\frac{dB}{dr} - E\right) \tag{5}$$

is the equilibrium angular drift velocity.

The expression in the last brackets of Eq. (4) is equal to the time derivative of $\hat{E}_r$ in a coordinate system which rotates with the angular velocity φ_0. Since the perturbation propagates by the motion of the ions, this is also the phase velocity of the perturbing wave (apart from a small deviation due to the velocity shear in the ring), and the expression would vanish (apart from a corresponding term of relative order a/R with respect to the first expression) if E_r were due to a wave which is neither damped nor growing. In the case of an unstable wave the expression may be neglected with respect to the other terms in Eq. (4) as long as the growth rate is smaller than $r\,d\dot{\varphi}_0/dr$. We shall henceforth make this assumption and, therefore, keep in mind that our formalism will not be valid for highly unstable situations, i. e. for cases where the above condition is not fulfilled.

The remaining terms in Eq. (4) are slowly varying functions of r, which change over characteristic lengths much larger than the minor radius of the plasma ring. The smallest characteristic length is that of $\hat{E}_\varphi$. As is shown in Ref. [9], it is of the order

$$L = a\left[(8/\sqrt{\pi})\log(2\,R/a)\right]^{1/2} \tag{6}$$

for $\lambda = 2\,\pi\,R$. In the domain of the plasma ring, i. e. for $R - a < r < R + a$, the slowly varying terms may therefore be approximated by their values for $r = R$. The resulting equation is an autonomous equation of motion for the azimuthal angle φ:

$$\ddot{\varphi} = \frac{e}{M\,R}\hat{E}_\varphi(R) \tag{7}$$

with

$$M^{-1} = \frac{c^2\,R}{e\,B}\left[1 + \frac{m\,c^2}{e\,R\,B}\frac{d}{dR}\left(\frac{R\,E}{B}\right)\right]\frac{d}{dR}\left[\frac{1}{B\,R}\left(\frac{\mu}{e}\frac{dB}{dR} - E\right)\right]. \tag{8}$$

As is seen from Eq. (7), the quantity M plays the role of an effective mass. We shall see later that it is negative when the system is unstable.

Equation (7) can be given a canonical form. Defining the new variables

$$x = R\,\varphi, \quad p = M\,R\,\dot{\varphi}, \quad F(x,t) = e\,\hat{E}_\varphi(R,\varphi,t), \tag{9}$$

we have

$$\frac{dp}{dt} = F(x,t), \quad \frac{dx}{dt} = \frac{p}{M}. \tag{10}$$

These equations derive from the following Hamiltonian:

$$\mathcal{H} = \frac{p^2}{2\,M} - \int^x F(x',t)\,dx'.$$

III. Dispersion Relation

By the projection procedure described above the problem has become one-dimensional. An ion is now characterized by its position x on the ring, the generalized momentum p, and the magnetic moment μ. We therefore introduce a distribution function $f(x,p,\mu;t)$ depending on these variables and time. When integrated over x, p, and μ, it is assumed to be normalized as to give the total number N of ions in the ring.

Assuming N to be constant in time, we may start from the continuity equation in the (x,p,μ)-space and use the canonical Eqs. (10) and the fact that due to assumption 4) μ is a constant of motion [4], i. e. $d\mu/dt = 0$. We then arrive at the following Vlasov equation:

$$\frac{\partial f}{\partial t} + \frac{p}{M(\mu)}\frac{\partial f}{\partial x} + F\frac{\partial f}{\partial p} = 0. \tag{11}$$

[6] C. E. NIELSEN et al., Proc. of the CERN Symp. on High Energy Accelerators, CERN, Geneva 1959.

[7] R. A. DORY, Oak-Ridge Semiannual Report, ORNL-3908, October 1965.

[8] R. W. LANDAU and V. K. NIEL, Phys. Fluids **9**, 2412 [1966].

[9] D. VOSLAMBER, EUR-CEA-FC-414 [1966].

In order to make the problem self-consistent, the electric force F has to be reexpressed in terms of the distribution function f. It is shown in Ref. [7] that for $a \ll R$ [assumption 3)] the following approximation holds

$$F(x, t) = -\frac{e^2}{\varepsilon} \int_{-\infty}^{\infty} \mathrm{d}p \int_0^{\infty} \mathrm{d}\mu \int_0^{2\pi R} \mathrm{d}x' \, \frac{f^{(1)}(x', p, \mu; t) \, \cos \xi(x, x')}{\mathrm{d}^2(x, x')} \quad (12)$$

with

$$d^2 \approx 4 \, R^2 \sin^2 \frac{x' - x}{2 \, R} + a^2,$$

$$\cos \xi \approx \frac{R}{d} \sin \left(\frac{x' - x}{R} \right).$$

Further, $f^{(1)}$ is defined from the equation

$$f(x, p, \mu; t) \equiv f^{(0)}(p, \mu) + f^{(1)}(x, p, \mu; t)$$

where $f^{(0)}(p, \mu)$ denotes the equilibrium part of the distribution function. $\varepsilon = 1 + (\bar{\omega}_{\mathrm{pi}}/\bar{\omega}_{\mathrm{c}})^2$ is the dielectric constant perpendicular to the magnetic field, referring mainly to the middle plasma which is surrounded by the ring, $\bar{\omega}_{\mathrm{pi}}$ and $\bar{\omega}_{\mathrm{c}}$ being mean values of the ion plasma frequency and cyclotron frequency, respectively.

Writing the distribution function as a Fourier series in x,

$$f^{(1)} = \sum_{l=-\infty}^{\infty} e^{ilx/R} \, \tilde{f}_l(p, \mu; t),$$

we get from (12)

$$F(x, t) = -\frac{i e^2}{\varepsilon R} \sum_{l=-\infty}^{\infty} \beta_l g_l(t) \, e^{ilx/R}, \quad (13)$$

where

$$g_l(t) = \int_{-\infty}^{\infty} \mathrm{d}p \int_0^{\infty} \mathrm{d}\mu \, \tilde{f}_l(p, \mu; t),$$

$$\beta_l = R^3 \int_0^{2\pi} \frac{\sin l \, \Phi \sin \Phi \, \mathrm{d}\Phi}{[4 \, R^2 \sin^2 (\Phi/2) + a^2]^{3/2}} \, .$$

For $l = 1, 2, 3$ we have

$$\beta_1 = 2 [\log (R/a) + O(1)],$$
$$\beta_2 = 2 [\beta_1 + O(1)],$$
$$\beta_3 = 3 [\beta_1 + O(1)].$$

For the special case $l = 1$ the $O(1)$-term has been calculated in Ref. [9] on using a more rigorous expression than the above integral for β_l. The result found there is

$$\beta_1 = 2 [\log (R/a) + 0.7] \, .$$

[10] L. D. LANDAU, J. Phys. USSR **10**, 25 [1946].

The dispersion relation may now be derived in the usual way [10] by linearising Eq. (11) with respect to $f^{(1)}$, making use of the Fourier representation (13), and taking the Laplace transform with respect to time. Denoting the Laplace variable by s, we obtain

$$1 - \frac{i e^2 \beta_l}{\varepsilon R} \int \int \mathrm{d}p \, \mathrm{d}\mu \, \frac{\partial f^{(0)}/\partial p}{s + i \, l \, p/R \, M(\mu)} = 0 \, . \quad (14)$$

For further evaluation of this equation we need an explicit expression for the zero order distribution function $f^{(0)}(p, \mu)$. It may be expressed in terms of the probability density $w_{\mathrm{r}}(r)$ for finding an ion at radius r, and the probability density $w_{\mathrm{T}}(T)$ for an ion to have the transverse energy T. The variables p and μ are the following functions of r and T:

$$p = M(r, T) \, R \, \dot{\varphi}_0(r), \qquad \mu = T/B(r),$$

where $M(r, T)$ depends on r and T only via μ. Using the inverse functions $r(p, \mu)$ and $T(p, \mu) = \mu B[r(p, \mu)]$, we have

$$f^{(0)}(p, \mu) = \qquad\qquad\qquad\qquad (15)$$
$$\frac{N}{2 \, \pi \, R} \, w_{\mathrm{r}}[r(p, \mu)] \, w_{\mathrm{T}}[\mu \, B[r(p, \mu)]] \, \frac{\partial (r, T)}{\partial (p, \mu)},$$

where it is assumed that position and energy are stochastically independent.

The functions w_{r} and w_{T} will now be approximated by step functions. Defining the function

$$\Delta_k(x) = \begin{cases} \dfrac{1}{2 \, k} & \text{for} \quad |x| < k, \\ 0 & \text{for} \quad |x| > k, \end{cases}$$

we write

$$w_{\mathrm{r}} = \Delta_a(r - R), \quad w_T(T) = \Delta_{\Delta T}(T - \overline{T}), \quad (16)$$

where $\overline{T}$ denotes some mean energy of the injected ions and ΔT measures the dispersion of the energies around their mean value.

In the two extreme cases considered in this paper, namely that the ion precession is either mainly due to the magnetic field gradient

$$[|(\mu/e) \, (\mathrm{d}B/\mathrm{d}R)| \gg |E(R)|]$$

or to the radial electric field

$$[|(\mu/e) \, (\mathrm{d}B/\mathrm{d}R)| \ll |E(R)|] \, ,$$

the generalized momentum p does not depend on T. The Jakobien determinant then simplifies to

$$\frac{\partial (r, T)}{\partial (p, \mu)} = \frac{\partial r}{\partial p} \, B[r(p)] \, . \quad (17)$$

On integrating by parts with respect to p in Eq. (14), inserting (15) with (16) and (17) into this equation, and transforming back from p and μ to the integration variables r and T, the dispersion relation becomes

$$1 + \frac{e^2 \beta_l l N}{8 \pi \varepsilon R^3 a \Delta T} \int\limits_{R-a}^{R+a} \mathrm{d}r \int\limits_{\bar{T}-\Delta T}^{\bar{T}+\Delta T} \frac{[M(r,T)]^{-1} \, \mathrm{d}T}{[s + i l p(r)/R M(r,T)]^2} = 0 \, .$$

Since the r-integration is only over the small interval $R - a < r < R + a$ we replace all functions of r by their Taylor expansions, using the zero order result for the numerator, but retaining the linear terms in the denominator because of its singular behaviour. The r-integration may then be carried out. Introducing the dimensionless variables

$$\sigma = s/i l \omega_{0\varphi} \, , \quad \tau = T/\bar{T} \, , \quad \delta = \Delta T/\bar{T} \, , \quad (18)$$

$$\alpha(\tau) = \frac{a}{\omega_{0\varphi}} \frac{\partial \omega(R, \tau \bar{T})}{\partial R} \, , \quad \gamma(\tau) = \frac{\omega_\varphi(R, \tau \bar{T})}{\omega_{0\varphi}} \, ,$$

$$k = - \beta_l N e^2/2 \pi \varepsilon l R^3 M_0 \omega_{0\varphi}^2 \, ,$$

where

$$\omega_\varphi(r, T) = \frac{p(r)}{R M(r, T)} \, ,$$

$$\omega_{0\varphi} = \omega_\varphi(R, \bar{T}) \, , \quad M_0 = M(R, \bar{T}) \, ,$$

we obtain

$$1 - \frac{k}{2\delta} \int\limits_{1-\delta}^{1+\delta} \frac{\gamma(\tau) \, \mathrm{d}\tau}{\alpha^2(\tau) - [\sigma + \gamma(\tau)]^2} = 0 \, . \quad (19)$$

If the relative energy spread δ is negligibly small, or if the equilibrium precession is mainly due to the radial electric field, we have $\gamma = 1$ and α independent of τ. The τ-integration in Eq. (19) is then trivial, and the solution of the dispersion relation is

$$\sigma = -1 \pm \sqrt{-k + \alpha^2} \, . \quad (20)$$

It corresponds to an unstable mode if $k > \alpha^2$. This can only ocur if the effective mass M contained in k is negative. The angular phase velocity of the unstable wave is then

$$\mathrm{Im}\{s\}/l = \omega_{0\varphi} \, \mathrm{Re}\{\sigma\} = - \omega_{0\varphi} \, .$$

Note that Eq. (20) shows similarities to the corresponding formula for flute instabilities. However, while for the latter the quantity k would be proportional to the density gradient, it is proportional to the inverse effective mass in Eq. (20), i. e. to a quantity which represents first and second radial derivatives of the equilibrium fields. Further, in the flute case the stabilizing term α^2 would have to be replaced by the constant $1/4$. Finally, the phase velocity would be $\omega_{0\varphi}/2$ instead of $\omega_{0\varphi}$. The latter fact might yield a means to distinguish between the two instabilities experimentally in case they are not coupled.

IV. Finite Energy Spread

If the equilibrium precession is mainly caused by the magnetic field gradient, a finite energy spread may influence significantly the stability behaviour of the system. We have in this case $\gamma(\tau) = \tau$ and $\alpha = \alpha_0 \tau$ with $\alpha_0 = (a/\omega_{0\varphi}) (\partial \omega_{0\varphi}/\partial R)$. Further, the condition for passing from Eq. (4) to Eq. (7) leads to $k < 1$. This inequality is fulfilled in all cases of interest and is thus not a serious restriction. The dispersion relation (19) becomes

$$1 + k D(\sigma) = 0 \quad (21)$$

with

$$D(\sigma) = - \frac{1}{4 \alpha_0 \delta} \left[\frac{1}{1 + \alpha_0} \log \frac{(1 + \alpha_0)(1 + \delta) + \sigma}{(1 + \alpha_0)(1 - \delta) + \sigma} \right. \quad (22)$$

$$\left. - \frac{1}{1 - \alpha_0} \log \frac{(1 - \alpha_0)(1 + \delta) + \sigma}{(1 - \alpha_0)(1 - \delta) + \sigma} \right]$$

where the principal values of the logarithms are to be taken. $D(\sigma)$ is an analytical function in the σ-plane, which is cut on the real axis from $\sigma_1 = -(1 + \alpha_0)(1 + \delta)$ to $\sigma_2 = -(1 + \alpha_0)(1 - \delta)$ and from $\sigma_3 = -(1 - \alpha_0)(1 + \delta)$ to $\sigma_4 = -(1 - \alpha_0)(1 - \delta)$. From the Nyquist theorem it can be seen that Eq. (21) has two solutions. Further, if σ_0 is a solution, so is the conjugate complex σ_0^*.

In order to know under what conditions the solutions are real (i. e. stable or marginally stable) we calculate $D(\sigma)$ in the vicinity of the real axis of the σ-plane. Writing $\sigma = x + i y$, we get to first order in y

$$D = - \frac{1}{4 \alpha_0 \delta} \left[\frac{1}{1 + \alpha_0} \log \left| \frac{x - \sigma_1}{x - \sigma_2} \right| - \frac{1}{1 - \alpha_0} \log \left| \frac{x - \sigma_3}{x - \sigma_4} \right| \right]$$

$$+ \frac{i \pi}{4 \alpha_0 \delta} g(x) + \frac{i y}{2 \alpha_0} \left[\frac{1}{(x - \sigma_1)(x - \sigma_2)} - \frac{1}{(x - \sigma_3)(x - \sigma_4)} \right]$$

where

$$g(x) = \begin{cases} 0 & \text{if } x < \sigma_1 \text{ or } \sigma_2 < x < \sigma_3 \text{ or } \sigma_4 < x \, , \\[2mm] \pm \dfrac{1}{1 + \alpha_0} & \text{if } \sigma_1 < x < \min(\sigma_2, \sigma_3) \, , \\[2mm] \mp \dfrac{2 \alpha_0}{1 - \alpha_0^2} & \text{if } \min(\sigma_2, \sigma_3) < x < \sigma_2 \, , \\[2mm] \pm \dfrac{1}{1 - \alpha_0} & \text{if } \max(\sigma_2, \sigma_3) < x < \sigma_4 \, . \end{cases}$$

The upper and lower signs have to be taken at the upper and lower endges of the cuts of the real axis, respectively.

The shape of the real part of D on the x-axis is represented in Fig. 1 for a case where the two cuts do not overlap $(\sigma_2 < \sigma_3)$. Since $1/k$ has to be added
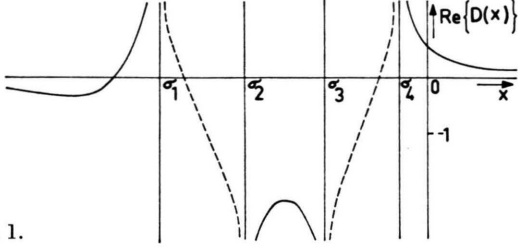

Fig. 1.

to D in the dispersion relation, the curve in Fig. 1 has to be lifted by this amount. (We henceforth assume $k > 0$ because negative k, i. e. positive effective masses M, lead to stable solutions anyway.) The zero on the left of σ_1 then disappears because $k < 1$. The zeros of the dotted parts of the curve need not be considered because in the intervals $\sigma_1 < x < \sigma_2$ and $\sigma_3 < x < \sigma_4$ the imaginary part of D does not vanish for $y \to 0$. Hence, these intervals do not contain zeros of the dispersion relation. On the other hand, we have to determine for each special choice of k whether or not the middle part of the curve lying between σ_2 and σ_3 cuts the x-axis. If it does, say at $x = x_{01}$ and $x = x_{02}$, we find $\sigma = x_{01}$ and $\sigma = x_{02}$ as real solutions of the dispersion relation. This means that the system is stable. If $\sigma_2 < \sigma_3$, i. e. $\delta < a_0$, this situation can always be achieved by choosing k (representing essentially the density) sufficiently small. On the other hand, if $\sigma_2 > \sigma_3$ (i. e. $\delta > a_0$), the middle part of the curve does not exist and there is no stable solution of the dispersion relation. The two existing solutions are then non-real and the system is unstable.

A more quantitative study can be done for small δ (say $\delta \lesssim a_0$). On writing

$$(1 + a_0)^{-1} \log u - (1 - a_0)^{-1} \log v$$
$$= (1 - a_0^2)^{-1} \log[u^{1-a_0}/v^{1+a_0}]$$

in Eq. (22) and expanding in powers of δ, we obtain

$$D(\sigma) =$$
$$\frac{1}{4 a_0 \delta (1 - a_0^2)} \log \frac{(\sigma+1)^2 - [a_0 - \delta(1 - a_0^2)]^2 + O(a_0 \delta^2)}{(\sigma+1)^2 - [a_0 + \delta(1 - a_0^2)]^2 + O(a_0 \delta^2)} .$$

The O-terms have been determined on using the fact that $\sigma \approx -1$ for solutions of the dispersion relation. Neglecting these terms, as well as terms of order $a_0^3 \delta$, leads to the following solution:

$$\sigma \approx -1 \pm \left[\frac{(a_0 - \delta)^2 - (a_0 + \delta)^2 \exp(-4 a_0 \delta/k)}{1 - \exp(-4 a_0 \delta/k)} \right]^{1/2} .$$
(23)

This approximation becomes invalid for $\delta \gtrsim a_0$, however, for $\delta = a_0$ it still yields the result that the system is unstable for all values of k (i. e. even for arbitrarily small densities). This is in accordance with the statement made above in the discussion of Fig. 1. One also sees that for $\delta < a_0$ the system is stable for sufficiently small k, i. e. for sufficiently small density.

Since Eq. (23) becomes exact in the limit of very small δ, it is able to describe the effect of a small energy spread δ on an equilibrium with originally $\delta = 0$. For this purpose we expand the expression under the square root for $4 a_0 \delta \ll k$ and obtain

$$\sigma = -1 \pm \sqrt{-k + a_0^2 + \delta^2(1 - 4 a_0^2/3 k)} . \quad (24)$$

According to this formula a small energy spread is destabilizing if $k < 4 a_0^2/3$, and stabilizing if $k > 4 a_0^2/3$. If for $\delta = 0$ the system is stable $(k < a_0^2)$ it becomes less stable for $\delta \neq 0$; if it is marginally stable $(k = a_0^2)$ it becomes unstable. On the other hand, if the system is "quite unstable" $(k > 4 a_0^2/3)$ it becomes less unstable for $\delta \neq 0$.

So far our quantitative results are restricted to small δ. Now, an extension to the whole domain $0 < \delta < 1$ can be obtained easily if one fixes $a_0 = 1/3$, a very realistic value from the experimental standpoint. The dispersion relation (21) then yields a cubic equation. One of the three solutions must be excluded because it does not satisfy the original Eq. (21). The other two solutions are

$$\sigma = -\tfrac{8}{9} - \tfrac{1}{3} \left\{ [\eta + (\xi^3 + \eta^2)^{1/2}]^{1/3} + [\eta - (\xi^3 + \eta^2)^{1/2}]^{1/3} \right\}$$
$$\pm \frac{i}{\sqrt{3}} \left\{ [\eta + (\xi^3 + \eta^2)^{1/2}]^{1/3} - [\eta - (\xi^3 + \eta^2)^{1/2}]^{1/3} \right\}$$
(25)

where

$$\eta = -\tfrac{1}{27} + (\tfrac{1}{3} + \delta^2) H - \delta^2, \quad \xi = -\tfrac{1}{9} + \tfrac{2}{3} H - \delta^2$$

and

$$H = \delta \frac{1 + \exp(-16 \delta/9 k)}{1 - \exp(-16 \delta/9 k)} .$$

Numerical values for σ have been computed from Eq. (25) for some values of k and δ. The results are listed in the table below, where stable and mar-

ginally stable cases are denoted by "stab." and "marg.", respectively, and the unstable cases are characterized by their growth rates $\text{Im}\{\sigma\}$.

k ╲ δ	0	1/10	1/3	1
1/25	stab.	stab.	0.003	0.000
1/9	marg.	0.06	0.09	0.000
2/5	0.54	0.53	0.46	0.14

Table 1.

Let us now try to understand the physical reason for the somewhat surprising result that an energy spread may have a destabilizing effect. Suppose for a moment that the electric field of the perturbation is switched off, so that the latter moves by "free streaming" (i. e. by its free precession in the equilibrium field). In the case of zero energy spread the angular precession velocity is a function of radius only; it therefore has a shear which leads to disperse the initial perturbation after some time. In the case of a finite energy spread, however, the precession velocity also depends on the transverse energy. Consider then two ions at the same angular position but at different radii in the plasma ring. If they had the same energy (zero energy spread) they would move away from each other, but due to a finite energy spread they may have energies whose difference just compensates the difference of the radii in such a way as to yield equal angular precession velocities. Hence, a fraction of the particles which constitute the perturbation do not have a velocity shear and do not decay by free streaming. In the presence of the perturbing electric fields this part of the perturbation will grow, provided that it is extended over a sufficient range of radii in the plasma ring. To guarantee this, the energy spread must not be too small. On the other hand, for very large energy spreads only a small fraction of the particles will be available for compensating the velocity shear, while most of the particles spread out, thus destroying a large part of the perturbation. This is why the growth rates decrease rapidly toward zero as the relative energy spread δ tends to one (see Table 1).

V. Finite Larmor Radius

Assumption 4) involves a severe restriction for most practical applications. It would therefore be desirable to estimate the influence of moderately

finite Larmor radii on our results. Modifications are expected for two reasons:
1) The equation of motion for the guiding centers of the ions will be modified due to the inhomogeneities of the electro-magnetic field.
2) The expression for the electric field will modify due to the difference between the particle and the guiding center densities.

To estimate the first effect, it is sufficient to correct only the dominant terms in Eq. (1), and to consider only the electric field because the lengths characterizing its inhomogeneity (a for E_r and L for $\hat{E}_\varphi$ [see Eq. (6)]) are small compared to the corresponding length R for the magnetic field. We therefore correct the $c\,\boldsymbol{E} \times \boldsymbol{B}/B^2$ term, which, following a method described in Ref. [11] becomes approximately

$$\boldsymbol{v}_{\mathrm{E}} = -\,c\,\frac{\boldsymbol{B}}{B^2} \times \Big(1 + \frac{\varrho^2}{4}\,\nabla_{\perp}{}^2\Big)\boldsymbol{E}\,, \qquad (26)$$

where $\nabla_\perp$ is the two-dimensional nabla operator in the plane perpendicular to $\boldsymbol{B}$.

Corresponding to the second effect the electric field in this equation has to be calculated from the particle density ν which is related to the guiding center density ν_g by a similar equation [11]:

$$\nu = \Big(1 + \frac{\varrho^2}{4}\,\nabla^2{}_\perp\Big)\nu_g\,. \qquad (27)$$

Of course, Eqs. (26) and (27) are only valid if the relative corrections containing ϱ^2 are small. The most pessimistic situation occurs when the equilibrium precession is caused by a radial electric field which is mainly built up by an ion excess in the plasma ring itself. An estimate of the corrections in Eqs. (26) and (27) can be made using a plane model (corresponding to $R = \infty$). Assuming a plasma sheath with an ion density excess proportional to, say, $\exp[-(r-R)^2/a^2]$, one finds for the relative correction of the drift velocity the expression

$$-\varrho^2\,x\,e^{-x^2}/2\,a^2 \int_0^x e^{-\xi^2}\,\mathrm{d}\xi \quad [x = (r-R)/a] \quad (28)$$

with a maximum value of $\varrho^2/2\,a^2$ for $r = R$. To this correction one has to add that from Eq. (27) which turns out to be equal to the expression (28) including the sign. Hence, the total maximal relative correction is $(\varrho/a)^2$. In order that this quantity be

[11] G. SCHMIDT, Physics of High Temperature Plasmas, Academic Press, New York 1966.

smaller than unity, the Larmor radius must stay below the minor radius of the plasma ring.

In cases where the equilibrium precession is primarily due to the magnetic field gradient, the estimates are more optimistic. The smallest characteristic inhomogeneity length involved in the formalism is then that of $\hat{E}_\varphi$. For $l = 1$ its order is that given in Eq. (6). A detailed analysis [9] shows that an estimate of the final correction in the dispersion relation may be obtained by multiplying the quantity k with $1 - l \sqrt{\pi} \overline{\varrho^2}/4 \beta_l a^2$, where $\overline{\varrho^2}$ denotes the mean square of the ionic Larmor radius. This result is valid under the condition $l^2 \ll (R/a)^2$. It shows a stabilizing effect of the finite Larmor radius, but indicates that the effect is not significant if the mean Larmor radius does not exceed too much the minor radius of the plasma ring.

VI. Applications

As a first example we consider the M.M.I.I. device, where our validity criteria are moderately fulfilled. The equilibrium precession is there mainly caused by a strong magnetic field gradient. The dependence of the magnetic field on radius is well represented by the parabola $B = B_0[1 - (r/16\ \mathrm{cm})^2]$ with $B_0 = 11\,000$ Gauss. The radial profile of the guiding center density has a steep relative maximum at $R = 9$ cm with a half width of about 2 cm. The ions have a mean energy of 35 keV with a relative spread of at most $2\,\delta = 1/3$ (using a magnetron ion source). For $l = 1$ one calculates with these data $k = 1.3 \cdot 10^{-10}\,\nu$ cm^3, $a_0^2 = 0.17$. According to formula (24) the system becomes unstable above the critical density $\nu_c \approx 10^9$ cm^{-3}. The maximum growth rate (for large densities) is about $3 \cdot 10^6$ sec^{-1}. The precession frequency at $R \approx 9$ cm is $\omega_{0\varphi} = -5.3 \cdot 10^6$ sec^{-1}.

The critical density for flute instabilities is also expected to be of order 10^9 cm^{-3}. Hence, both instabilities will be in competition and a more complete theoretical study should treat them simultaneously.

To close this paper we consider a situation where the radial electric field is the dominant cause for the equilibrium precession. Under certain operating conditions this is the case in Ogra II, but the roughness of our model makes the applicability of our formalism somewhat doubtful.

On neglecting the inhomogeneity of the magnetic field with respect to that of the electric field, Eq. (8) yields the following approximate expression for the inverse effective mass:

$$M^{-1} \approx -\frac{c^2}{e\,B^2}\left[1 + \frac{m\,c^2}{e\,B^2}\frac{\mathrm{d}E}{\mathrm{d}R}\right]\frac{\mathrm{d}E}{\mathrm{d}R}.$$

Using Poisson's equation, we approximate $\mathrm{d}E/\mathrm{d}R$ by $4\,\pi\,\gamma\,\nu_i\,e$, where $\gamma \equiv (\nu_i - \nu_e)/\nu_i$ denotes the relative ion density excess. This leads to

$$M \approx -m\,\omega_c^2/\gamma\,\omega_{pi}^2(1 + \gamma\,\omega_{pi}^2/\omega_c^2),$$

and thus, since the second term in the denominator is negligible, to

$$k = (\beta_l\gamma/4\,l)\,(a/R)^2\,(\omega_{pi}^4/\omega_c^2\,\omega_{0\varphi}^2).$$

In the same manner we find

$$a = -\gamma\,(a/R)\,\omega_{pi}^2/\omega_c\,\omega_{0\varphi}.$$

Hence, for $k > a^2$, Eq. (20) yields the following growth rate:

$$\omega_{0\varphi}\,\mathrm{Im}\{\sigma\} = \frac{a}{R}\frac{\omega_{pi}^2}{\omega_c}\sqrt{\gamma^2 - \gamma\,\beta_l/4\,l}.$$

For $l = 1$ the expression under the square root becomes $\gamma[\gamma - (1/2)\log(R/a) - 0.35]$. In the idealized model we have $R \gg a$, and the negative contribution would always prevail and yield instability. In practice, however, the logarithm is not much larger than one. Further, allowing for finite Larmor radius effects, the negative contribution in the brackets may still be reduced and may be less than one. However, since $\gamma < 1$, the total expression may be negative for sufficiently small ion density excess and thus lead to instability. On the other hand it should be noted that the growth rate is very small because of the factor $(a/R)\,(\omega_{pi}^2/\omega_c)$. Therefore, it will at most be observable in the absence of flute instabilities, i. e. below the critical density of the latter.